

rings in which power values of K -Engels with derivations annihilate a certain element

shervin sahebi¹ and venus rahmani²

Department of Mathematics, Islamic Azad University,
Tehran Centre, Tehran, IRAN

e-mail¹: sahebi@iauctb.ac.ir

e-mail²: ven.rahmani.math@iauctb.ac.ir

Abstract

let R be a 2-torsion free semiprime ring and d a non-zero derivation. Further let $A = O(R)$ be the orthogonal completion of R and $B = B(C)$ the Boolean ring of C where C be the extended centroid of R . We show that if $a[[d(x), x]_n, [y, d(y)]_m]^t = 0$ such that $0 \neq a \in R$ for all $x, y \in R$, where $m, n, t > 0$ are fixed integers, then there exists an idempotent $e \in B$ such that eA is a commutative ring and d induce a zero derivation on $(1 - e)A$.

Math. Subj. Classification 2010: 16R50; 16N60; 16D60.

Key Words: prime ring, semiprime ring, derivation.

1. Introduction

Let R be an associative ring with center $Z(R)$. Recall that an additive mapping d of R into itself is a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Also if $(x_i)_{i \in \mathbb{N}}$ is a squence of elements of R and k is a positive integer, we define $[x_1, \dots, x_{k+1}]$ inductively as follows:

$$[x_1, x_2] = x_1x_2 - x_2x_1 \quad , \quad [x_1, \dots, x_k, x_{k+1}] = [[x_1, \dots, x_k], x_{k+1}].$$

If $x_1 = x$ and $x_2 = \dots = x_{k+1} = y$, the notation $[x, y]_k$ is used to denote $[x_1, \dots, x_{k+1}]$ and $[x, y]_k$ is called a k -Engel element.

A well known result of Posner stated that if $[[d(x), x], y] = 0$ for all $x, y \in R$, then R is commutative [11]. A number of authors extended this result in several ways. Bell and Martindale in [2] studied this identity for a semiprime ring R . They proved that if R is a semiprime ring and $[[d(x), x], y] = 0$ for all x in a non-zero left ideal of R and $y \in R$, then R contains a non-zero central ideal. In [6], Filippis showed that if R is a prime ring with $\text{char} R \neq 2$ and d a non-zero derivation of R such that $[[d(x), x], [d(y), y]] = 0$ for all $x, y \in R$, then R is commutative. Recently Dhara obtained results for a prime ring R of $\text{char} R \neq 2$, with a nonzero derivation d that if $0 \neq a \in R$ such that $a[[d(x), x]_n, [d(y), y]_m] = 0$ for all $x, y \in R$, where $m, n \geq 0$ are fixed integers, then R is commutative [4]. Now, we will generalize Posner's result [11] when the condition are more widespread.

The main result of this paper is as follows:

Theorem 1.1. *let R be a 2-torsion free semiprime ring with non-zero derivation d and $0 \neq a \in R$ such that $a[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in R$, where $m, n, t > 0$ are fixed integers. Further let $A = O(R)$ be the orthogonal completion of R and $B = B(C)$ where C the extended centroid of R . Then there exists an idempotent $e \in B$ such that eA is a commutative ring and d induce a zero derivation on $(1 - e)A$.*

Throughout the paper we use the standard notation from [1]. In particular, we denote by Q the two sided Martindale quotient of prime and semiprime ring R and C the center of Q . We call C the extended centroid of R . It is well known that any derivation of prime(semiprime) ring R can be uniquely extended to a derivation of Q , and so any derivation of R can be defined on the whole of Q . Moreover Q is a prime(semiprime) ring as well as R . We refer to [1, 9] for more details.

2.Proof of main result

The following results are usefull tool needed the proof of main result.

Theorem 2.1. *Let R be a prime ring of $\text{char} R \neq 2$ and d a derivation of R . Suppose $a[[d(x), x]_n, [d(y), y]_m]^t = 0$ and $0 \neq a \in R$ for all $x, y \in R$, where $m, n, t > 0$ are fixed integers. Then R is commutative or $d = 0$.*

Proof. Consider two cases.

case 1. d is not a Q -inner derivation. By Kharchenko's Theorem [7] for any $x, y, z, s \in R$ we have $a[[z, x]_n, [s, y]_m]^t = 0$. This is a polynomial identity and hence there exists a field F such that $R \subseteq M_k(F)$ with $k > 1$ and $R, M_k(F)$ satisfy the same polynomial identity [8]. Therefore we can consider $a = (a_{ij})_{k \times k}$. We may assume that t is an even integer. Now putting $z = e_{ij}$, $x = e_{ii}$, $s = e_{ji}$, $y = e_{ii}$. Thus for any $i \neq j$, we have

$$0 = a[[z, x]_n, [s, y]_m]^t = a(-1)^{nt}(e_{ii} + (-1)^t e_{jj}) = a(e_{ii} + e_{jj}),$$

This implies $a_{ij} = 0$ for any i, j ($i \neq j$), which is contradiction.

case 2. d is a Q -inner derivation. So there exists an element $b \in Q$ such that $d(x) = [b, x]$ for all $x \in R$. Since by [3] Q and R satisfy the same generalized polynomial identities (GPI), hence for any $x, y \in Q$ we have $a[[b, x]_{n+1}, [y, [b, y]]_m]^t = 0$. Also since Q remains prime by the primeness of R , replacing R by Q we may assume that $b \in R$ and the extended centroid of R is just the center of R . Note that R is a centrally closed prime C -algebra in the present situation [5]. If R is commutative, we have nothing to prove. So, let R be noncommutative. Therefore R satisfies a nontrivial (GPI). Since R is a centrally closed prime C -algebra, by Martindale's Theorem [10], R is a strongly primitive ring. Let ${}_R V$ be a faithful irreducible left R -module with commuting ring $D = \text{End}({}_R V)$. By the Density Theorem, R acts densely on V_D . For any given $v \in V$ we claim that v and bv are D -dependent. Assume first that $av \neq 0$. Suppose on the contrary that v and bv are D -independent.

If $b^2v \in \text{span}\{v, bv\}$, then $b^2v = v\alpha + bv\beta$ for some $\alpha, \beta \in D$. By density of R in $\text{End}(V_D)$ there exist two elements x and y in R such that $xv = v$, $xbv = 0$ and $yv = 0$, $ybv = v$. Then

$$0 = a[[b, x]_{n+1}, [y, [b, y]]_m]^t v = (-2)^{mt} av.$$

If $b^2v \notin \text{span}\{v, bv\}$, then $\{v, bv, b^2v\}$ are all D -independent. Then by Density of R in $\text{End}(V_D)$ there exist two elements x and y in R such that $xv = v$, $xbv = 0$, $xb^2v = 0$ and $yv = 0$, $ybv = 0$, $yb^2v = 0$. Therefore we have

$$0 = a[[b, x]_{n+1}, [y, [b, y]]_m]^t v = (-2)^{mt} av.$$

Since $\text{char} R \neq 2$ we get $av = 0$, a contradiction. Thus v and bv are D -dependent as claimed. Assume next that $av \neq 0$. Since $a \neq 0$, we have $aw \neq 0$ for some $w \in V$. Then $a(v + w) = aw \neq 0$. Applying the first situation we have $bw = w\alpha$ and $b(v + w) = (v + w)\beta$, for some $\alpha, \beta \in D$. But v and w are clearly D -independent, and so there exist two elements x and y in R such that $xw = w$, $xv = 0$ and $yw = v$, $yv = 0$. Then

$$0 = a[[b, x]_{n+1}, [y, [b, y]]_m]^t = (-1)^{t(n+1)} 2^{mt} a(\beta - \alpha)^{2t} w,$$

which implies $\alpha = \beta$ and hence $bv = v\alpha$ as claimed. From the above we have proved that $bv = v\alpha(v)$ for all $v \in V$, where $\alpha(v) \in D$ depends on $v \in V$. In fact, it is easy to check that $\alpha(v)$ is independent of the choice of $v \in V$. That is, there exist $\delta \in D$ such that $bv = v\delta$ for all $v \in V$. we claim $\delta \in Z(D)$, the center of D . Indeed, if $\beta \in D$, then $b(v\beta) = (v\beta)\delta = v(\beta\delta)$ and the other hand $b(v\beta) = (bv)\beta = (v\delta)\beta = v(\delta\beta)$. Therefore $v(\beta\delta - \delta\beta) = 0$ so $\beta\delta = \delta\beta$, which implies $\delta \in Z(D)$. Thus $b \in C$ and hence $d = 0$, as be wanted.

The following example shows the hypothesis of primeness is essential in Theorem 2.1.

example 2.2. Let S be any ring, and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\}$.

Define $d : R \rightarrow R$ as follows: $d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then d is a non-zero derivation of R such that $a[[d(x), x]_n, [d(y), y]_m]^t = 0$ for all $x, y \in R$, where $m, n, t > 0$ are fixed integers, however R is not commutative.

Now let R be a semiprime orthogonally complete ring with extended centroid C . We use the notation $B = B(C)$ and $\text{spec}(B)$ to denote Boolean ring of C and the set of all maximal ideal of B . It is well known that if $M \in \text{spec}(B)$ then $R_M = R/RM$ is prime [1, Theorem 3.2.7]. We refer to [1, pages 37, 38, 43, 120] for definations of Ω - Δ -ring, a first order formula of signature Ω - Δ , Horn formulas and Hereditary first order formulas.

In preparation for the proof of Theorem we have the following lemma.

lemma 2.3.[1, Theorem 3.2.18]. Let R be an orthogonally complete Ω - Δ -ring with extended centroid C , $\Psi_i(x_1, x_2, \dots, x_n)$ Horn formulas of signature Ω - Δ , $i = 1, 2, \dots$ and $\Phi(y_1, y_2, \dots, y_m)$ a Hereditary first order formula such that $\neg\Phi$ is a Horn formula. Further, let $\vec{a} = (a_1, a_2, \dots, a_n) \in R^{(n)}$, $\vec{c} = (c_1, c_2, \dots, c_m) \in R^{(m)}$. Suppose that $R \models \Phi(\vec{c})$ and for every $M \in \text{spect}(B)$ there exists a natural number $i = i(M) > 0$ such that

$$R_M \models \Phi(\phi_M(\vec{c})) \implies \Psi_i(\phi_M(\vec{a})),$$

where $\Phi_M : R \rightarrow R_M = R/RM$ is the canonical projection. Then there exist a natural number $k > 0$ and pairwise orthogonal idempotents $e_1, e_2, \dots, e_k \in B$ such that $e_1 + e_2 + \dots + e_k = 1$ and $e_i R \models \Psi_i(e_i \vec{a})$ for all $e_i \neq 0$.

Denote by $O(R)$ the orthogonal completion of R which is defined as the intersection of all orthogonally complete subset of \mathcal{Q} containing R . Now

we can prove Theorem 1.1.

Proof of Theorem 1.1. It is well known that the derivation d can be extended uniquely to a derivation $d : Q \rightarrow Q$. According to [1, Theorem 3.1.16] $d(A) \subseteq A$ and $d(e) = 0$ for all $e \in B$. Therefore A is an orthogonally complete Ω - Δ -ring where $\Omega = \{o, +, -, \cdot, d\}$. Consider formulas

$$\Phi = (\exists a \neq 0)(\forall x)(\forall y)\|a[[d(x), x]_n, [y, d(y)]_m]^t = 0\|,$$

$$\Psi_1 = (\forall x)(\forall y)\|xy = yx\|,$$

$$\Psi_2 = (\forall x)\|d(x) = 0\|.$$

One can easily check that Φ is a hereditary first order formula and $\neg\Phi$, Ψ_1 , Ψ_2 are Horn formulas. So using Theorem 2.1 shows that all conditions of Lemma 2.3 are fulfilled. Hence there exist two orthogonal idempotent e_1 and e_2 such that $e_1 + e_2 = 1$ and if $e_i \neq 0$, then $e_i A \models \Psi_i$, $i = 1, 2$. The proof is complete. \square

References

- [1] K. I. Beidar, W. S. Martindale and A. V. Mikhalev, *Rings with generalized identities*, *Pure and Applied math*, (New york: Deloker)
- [2] H. E. Bell and W. S. Martindale III, *Centralizing mappings of semiprime rings*, *Canadian Mathematical Bulletin*, **30** (1987), 92–101.
- [3] C. L. Chung, *GPIs having coefficients in Utumi quotient rings*, *proc. Amer. Math. soc*, **103**(1988), 723–728.
- [4] B. Dhara, *On the annihilators of derivations with Engel conditions in prime rings*, *Tamsui oxford jurnal of Mathematical Sciencs*, **26**(3) (2010) 255–264.

- [5] J. S. Ericson, W.S. Martindale 3rd, and J.M. Osborn, *prime nonassociative algebras, pacific J. math*, **60** (1975), 49–63.
- [6] V. De. Filippis, *On derivations and commutativity in prime rings*, *Int.J. Math.Sci*, **70** (2004), 3859–3856.
- [7] V. K. Kharchenko, *Differential identity of prime rings*, *Algebra and Logic*, **17** (1978), 155–168.
- [8] C. Lanski, *An Engel condition with derivation*, *Proc. Amer. Math. Soc*, **118** (1993), 731–734.
- [9] T. K. Lee, *Semiprime rings with differential identities*, *Bull. Inst. Math. Acad. Sinica*, **20** (1992), 27–38.
- [10] W.S. Martindale III, *prime rings satistying a generalized polynomial identity*, *J.Algebra*, **12**(1969), 576–584.
- [11] E. C. Posner, *derivation in prime rings*, *Proc. Amer. Math. Soc*, **8** (1957), 1093–1100.